Lift of noninvariant solutions of heavenly equations from three to four dimensions and new ultra-hyperbolic metrics

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Abstract

We demonstrate that partner symmetries provide a lift of noninvariant solutions of three-dimensional Boyer-Finley equation to noninvariant solutions of four-dimensional hyperbolic complex Monge-Ampère equation. The lift is applied to noninvariant solutions of the Boyer-Finley equation, obtained earlier by the method of group foliation, to yield noninvariant solutions of the hyperbolic complex Monge-Ampère equation. Using these solutions we construct new Ricci-flat ultra-hyperbolic metrics with non-zero curvature tensor that have no Killing vectors.

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1 Introduction

In his paper [1] Plebañski introduced his first and second heavenly equations for a single potential governing Ricci-flat metrics on 4-dimensional complex manifolds. Solutions of these equations determine (anti-)self-dual heavenly metrics which satisfy the complex vacuum Einstein equations. There are two real cross sections of the complex metrics governed by the first heavenly equation, namely Kähler metrics with Euclidean or ultra-hyperbolic signature. The first heavenly equation in these cases coincides with the elliptic and hyperbolic complex Monge-Ampère equation (CMA) respectively that have applications to important problems in physics and geometry. In particular,

some solutions $u = u(z_1, \bar{z}_1, z_2, \bar{z}_2)$ to the elliptic CMA

$$u_{1\bar{1}} u_{2\bar{2}} - u_{1\bar{2}} u_{2\bar{1}} = 1 \tag{1.1}$$

can be interpreted as gravitational instantons. (From now on subscripts denote partial derivatives with respect to corresponding variables and bars mean complex conjugates.) The most important gravitational instanton is the Kummer surface $K3$ [2]. The explicit construction of $K3$ metric is still an unsolved challenging problem. One of the basic difficulties is that the metric should have no Killing vectors and hence the corresponding solution of CMA should have no symmetries, i.e. be a noninvariant solution. That means that the traditional method of Lie symmetry reduction cannot be applied for finding solutions of the heavenly equations and therefore there is a problem of finding their noninvariant solutions. We have recently developed the method of partner symmetries appropriate for this problem and obtained certain classes of noninvariant solutions to the elliptic and hyperbolic CMA and second heavenly equation together with corresponding heavenly metrics with no Killing vectors [3–5].

In this paper we obtain new noninvariant solutions of the four-dimensional hyperbolic complex Monge-Ampère equation ($HCMA$)

$$u_{1\bar{1}} u_{2\bar{2}} - u_{1\bar{2}} u_{2\bar{1}} = -1 \tag{1.2}$$

by lifting noninvariant solutions of the three-dimensional Boyer-Finley equation [6] to a four-dimensional solution manifold of $HCMA$. Noninvariant solutions to the elliptic Boyer-Finley equation were obtained first by D. Calderbank and P. Tod [7] and later, independently, in [8] where we had also proved non-invariance of these solutions. Here, for lifting to the solution manifold of $HCMA$, we use noninvariant solutions to the hyperbolic version of the Boyer-Finley equation that we have obtained in [8] by our version of the method of group foliation [9]. Using these solutions, we construct explicitly metrics with ultra-hyperbolic signature that have no Killing vectors. We hope that by an appropriate modification of this method we shall be able to obtain noninvariant solutions of the elliptic CMA and the corresponding Ricci-flat metrics with Euclidean signature and no Killing vectors. The other possibility is to obtain such metrics by a suitable analytic continuation directly from our ultra-hyperbolic metrics. A survey of results on four-dimensional anti-self-dual metrics with the ultra-hyperbolic signature was given by M. Dunajski in [10].

In section 2, for the sake of completeness, we show how a rotational symmetry reduction of $HCMA$, combined with a point and Legendre transformations, yields the Boyer-Finley equation.

In section 3 we introduce partner symmetries for the hyperbolic and elliptic complex Monge-Ampère equations and derive simplified equations for the case when the two partner symmetries coincide. This is possible only for the hyperbolic CMA.

In section 4 we apply a Legendre transformation combined with a simple point transformation, similar to the one in section 2 but with no symmetry reduction, to $HCMA$ and equations for partner symmetries. We show that, with the choice of rotational partner symmetries, a certain linear combination of $HCMA$ and two independent differential constraints resulting from this choice coincides with the hyperbolic version of the Boyer-Finley equation. Moreover, the two constraints, taken by themselves, yield the Bäcklund transformations for the Boyer-Finley equation that we had discovered earlier [11].

In section 5 we use this fact for lifting noninvariant solutions of the Boyer-Finley equation, that we had obtained in [8], to solutions of the $HCMA$ equation. Up to arbitrary symmetry transformations, we give a complete list of solutions of $HCMA$ that can be obtained by a lift from our noninvariant solutions of the Boyer-Finley equation.

In section 6 we show that our solutions of $HCMA$ are generically noninvariant. This means that, apart from a very special choice of arbitrary functions in these solutions, there is no symmetry of $HCMA$ with respect to which these solutions will be invariant. We present an explicit check of the non-invariance for the simplest one of our solutions.

In section 7 we consider four-dimensional hyper-Kähler metrics with the ultra-hyperbolic signature that are (anti-)self-dual solutions of Einstein equations provided the metric potential satisfies $HCMA$. We introduce the tetrad of the Newman-Penrose moving co-frame that provides an easiest and most convenient way to calculate Riemann curvature two-form. We apply the combination of a point and Legendre transformation, mentioned above, to the metric and moving co-frame, so that our exact solutions can serve as metric potentials for the transformed metric and moving co-frame.

Finally, in section 8 we use our solutions to obtain explicitly new ultra-hyperbolic metrics and the corresponding moving co-frames. Proceeding in a similar way to [3], one may check that since our solutions for the metric potentials are noninvariant, the resulting metrics have no Killing vectors. By

utilizing the moving co-frames, we were able to compute Riemann curvature two-forms for our solutions using the package EXCALC (Exterior Calculus of Modern Differential Geometry) [12] in the computer algebra system REDUCE 3.8 [13].

2 Rotational symmetry reduction of $HCMA$ to Boyer-Finley equation

The Boyer-Finley equation is obtained by a rotational symmetry reduction from the elliptic CMA [6]. A hyperbolic version of the Boyer-Finley equation appears as a result of symmetry reduction of $HCMA$ (1.2) with respect to the group of rotations in (x, y) plane or, in the complex coordinates $z_1 = x + iy$, rotations in the complex z_1-plane with the generator

$$X = y\partial_x - x\partial_y = i(\bar{z}_1\partial_{\bar{z}_1} - z_1\partial_{z_1})$$

and the symmetry characteristic [14] of the form $\varphi = i(z_1u_1 - \bar{z}_1u_{\bar{1}})$. Rotationally invariant solutions are determined by the condition $\varphi = 0$ which is satisfied by $u = u(r, z_2, \bar{z}_2)$ where $r = z_1\bar{z}_1 = x^2 + y^2$. For such u, depending only on three variables, $HCMA$ reduces to

$$(ru_{rr} + u_r)u_{2\bar{2}} - ru_{r2}u_{r\bar{2}} = -1.$$

Under the change of the invariant variable $\rho = \ln r = \ln z_1 + \ln \bar{z}_1$ the reduced equation becomes

$$u_{\rho\rho}u_{2\bar{2}} - u_{\rho 2}u_{\rho\bar{2}} = -e^{\rho}. \tag{2.1}$$

The Legendre transformation

$$u_\rho = p, \quad \rho = \phi_p, \quad u = p\phi_p - \phi, \quad z_2 = z, \quad \bar{z}_2 = \bar{z} \tag{2.2}$$

to the new unknown $\phi = \phi(p, z, \bar{z})$ takes the equation (2.1) to the form

$$\phi_{z\bar{z}} = e^{\phi_p}\phi_{pp} \equiv (e^{\phi_p})_p \tag{2.3}$$

that is related by $F = \phi_p$ to the hyperbolic version of the Boyer-Finley equation

$$F_{z\bar{z}} = (e^F)_{pp}.$$

3 Partner symmetries of complex Monge-Ampère equations

The determining equation for symmetries of $HCMA$ is the same as for the elliptic CMA (1.1)

$$\square(\varphi) = 0, \qquad \square = u_{2\bar{2}} D_1 D_{\bar{1}} + u_{1\bar{1}} D_2 D_{\bar{2}} - u_{2\bar{1}} D_1 D_{\bar{2}} - u_{1\bar{2}} D_2 D_{\bar{1}} \tag{3.1}$$

where φ denotes a symmetry characteristic and $D_i, D_{\bar{i}}$ are operators of the total derivatives with respect to $z^i, \bar{z}^i$ respectively. Therefore the construction of partner symmetries, given in this section, is the same for both elliptic and hyperbolic CMA [3, 4].

Define the operators

$$L_1 = \lambda(u_{1\bar{2}} D_{\bar{1}} - u_{1\bar{1}} D_{\bar{2}}), \quad L_2 = \lambda(u_{2\bar{2}} D_{\bar{1}} - u_{2\bar{1}} D_{\bar{2}}) \tag{3.2}$$

where λ is a complex constant. Then the operator $\square$ of the symmetry condition (3.1) can be expressed in terms of L_1, L_2 as $\square = \lambda^{-1}(D_1 L_2 - D_2 L_1)$. The symmetry condition takes the form of a total divergence

$$D_1 L_2 \varphi = D_2 L_1 \varphi \tag{3.3}$$

so that there locally exists a symmetry potential ψ defined by the differential equations

$$\psi_1 = L_1 \varphi = \lambda(u_{1\bar{2}} \varphi_{\bar{1}} - u_{1\bar{1}} \varphi_{\bar{2}}), \quad \psi_2 = L_2 \varphi = \lambda(u_{2\bar{2}} \varphi_{\bar{1}} - u_{2\bar{1}} \varphi_{\bar{2}}). \tag{3.4}$$

Because of the relation

$$[L_1, L_2] = \lambda^2 \{ (u_{1\bar{1}} u_{2\bar{2}} - u_{1\bar{2}} u_{2\bar{1}})_{\bar{1}} D_{\bar{2}} - (u_{1\bar{1}} u_{2\bar{2}} - u_{1\bar{2}} u_{2\bar{1}})_{\bar{2}} D_{\bar{1}} \}$$

the operators L_1 and L_2 commute on solution manifolds of the elliptic and hyperbolic CMA. Furthermore, we note the relation

$$D_1 L_2 - D_2 L_1 = L_2 D_1 - L_1 D_2. \tag{3.5}$$

Therefore, substituting φ by its potential ψ into the symmetry condition in the divergence form (3.3) and using the definition (3.4) and the relation (3.5), we obtain

$$D_1 L_2 \psi - D_2 L_1 \psi = L_2 \psi_1 - L_1 \psi_2 = [L_2, L_1] \psi = 0$$

and so the potential ψ of a symmetry φ is itself a symmetry. These two symmetries are called *partner symmetries*.

Now take complex conjugate to the equations (3.4) and solve them algebraically with respect to the φ_1 and φ_2, using elliptic or hyperbolic CMA, to obtain the inverse transformation

$$\varphi_1 = \mp\bar{\lambda}^{-1}(u_{1\bar{2}}\psi_{\bar{1}} - u_{1\bar{1}}\psi_{\bar{2}}), \quad \varphi_2 = \mp\bar{\lambda}^{-1}(u_{2\bar{2}}\psi_{\bar{1}} - u_{2\bar{1}}\psi_{\bar{2}}) \tag{3.6}$$

where the minus and plus signs correspond to the elliptic and hyperbolic CMA respectively. Note that for the $HCMA$ there is a simple possibility $\psi = \varphi$ when the equations (3.4) and (3.6) coincide and become

$$\varphi_1 = \lambda(u_{1\bar{2}}\varphi_{\bar{1}} - u_{1\bar{1}}\varphi_{\bar{2}}), \quad \varphi_2 = \lambda(u_{2\bar{2}}\varphi_{\bar{1}} - u_{2\bar{1}}\varphi_{\bar{2}}) \tag{3.7}$$

if $|\lambda| = 1$, i.e. $\lambda = e^{i\alpha}$ with a real α. For the elliptic CMA the ansatz $\psi = \varphi$ leads to a contradiction and no other obvious similar simplifications exist.

We will also need the equations, complex conjugate to (3.7)

$$\varphi_{\bar{1}} = \lambda^{-1}(u_{2\bar{1}}\varphi_1 - u_{1\bar{1}}\varphi_2), \quad \varphi_{\bar{2}} = \lambda^{-1}(u_{2\bar{2}}\varphi_1 - u_{1\bar{2}}\varphi_2). \tag{3.8}$$

We note that any three equations for partner symmetries out of the four ones (3.7) and (3.8) imply the fourth equation together with $HCMA$ itself as their algebraic consequences and, alternatively, the pair of first equations in (3.7) and (3.8) together with $HCMA$ imply the couple of second equations in these formulas. Thus, we have only three independent equations. For our future needs we choose $HCMA$ together with the first equations in (3.7) and (3.8)

$$\varphi_1 = \lambda(u_{1\bar{2}}\varphi_{\bar{1}} - u_{1\bar{1}}\varphi_{\bar{2}}), \quad \varphi_{\bar{1}} = \lambda^{-1}(u_{2\bar{1}}\varphi_1 - u_{1\bar{1}}\varphi_2) \tag{3.9}$$

as the basic independent equations.

4 Legendre transform of rotational partner symmetries and Boyer-Finley equation

Next we apply to $HCMA$ and the equations (3.9) the same combination of the point transformation and Legendre transformation, that produced the Boyer-Finley equation (2.3) in section 2 by the rotational symmetry reduction, but now we do not perform any symmetry reduction.

The point transformation $z_1 = e^{\zeta_1}$, $\bar{z}_1 = e^{\bar{\zeta}_1}$ yields $HCMA$ in the form

$$u_{\zeta_1\bar{\zeta}_1}u_{2\bar{2}} - u_{\zeta_1\bar{2}}u_{\bar{\zeta}_1 2} = -e^{\zeta_1+\bar{\zeta}_1} \tag{4.1}$$

and the partner symmetries equations (3.9) become

$$\begin{aligned}\varphi_{\zeta_1} &= \lambda e^{-\bar{\zeta}_1}(u_{\zeta_1\bar{2}}\varphi_{\bar{\zeta}_1} - u_{\zeta_1\bar{\zeta}_1}\varphi_{\bar{\zeta}_2})\\ \varphi_{\bar{\zeta}_1} &= \lambda^{-1}e^{-\zeta_1}(u_{2\bar{\zeta}_1}\varphi_{\zeta_1} - u_{\zeta_1\bar{\zeta}_1}\varphi_2).\end{aligned} \tag{4.2}$$

The Legendre transformation in the first pair of variables $\zeta_1, \bar{\zeta}_1$

$$\zeta_1 = \psi_q,\ \bar{\zeta}_1 = \psi_{\bar{q}},\ u = q\psi_q + \bar{q}\psi_{\bar{q}} - \psi,\ u_{\zeta_1} = q,\ u_{\bar{\zeta}_1} = \bar{q}, \tag{4.3}$$

with $z_2 = z, \bar{z}_2 = \bar{z}$, maps the unknown $u(\zeta_1, \bar{\zeta}_1, z_2, \bar{z}_2)$ to the new unknown $\psi(q, \bar{q}, z, \bar{z})$ and the symmetry characteristic transforms as $\varphi(\zeta_1, \bar{\zeta}_1, z_2, \bar{z}_2) = \Phi(q, \bar{q}, z, \bar{z})$. The inverse transformation is

$$q = u_{\zeta_1},\ \bar{q} = u_{\bar{\zeta}_1},\ \psi = \zeta_1 u_{\zeta_1} + \bar{\zeta}_1 u_{\bar{\zeta}_1} - u,\ \psi_q = \zeta_1,\ \psi_{\bar{q}} = \bar{\zeta}_1. \tag{4.4}$$

Under this transformation $HCMA$ (4.1) becomes

$$\psi_{q\bar{q}}\psi_{z\bar{z}} - \psi_{q\bar{z}}\psi_{\bar{q}z} = e^{\psi_q+\psi_{\bar{q}}}(\psi_{q\bar{q}}^2 - \psi_{qq}\psi_{\bar{q}\bar{q}}) \tag{4.5}$$

and the partner symmetries constraints (4.2) take the form

$$\begin{aligned}e^{\psi_{\bar{q}}}(\psi_{\bar{q}\bar{q}}\Phi_q - \psi_{q\bar{q}}\Phi_{\bar{q}}) &= \lambda(\psi_{q\bar{q}}\Phi_z - \psi_{q\bar{z}}\Phi_{\bar{q}})\\ e^{\psi_q}(\psi_{qq}\Phi_{\bar{q}} - \psi_{q\bar{q}}\Phi_q) &= \lambda^{-1}(\psi_{q\bar{q}}\Phi_{\bar{z}} - \psi_{\bar{q}z}\Phi_q).\end{aligned} \tag{4.6}$$

We use here the rotational symmetry characteristic $\varphi = i(z_1u_1 - \bar{z}_1u_{\bar{1}}) = i(u_{\zeta_1} - u_{\bar{\zeta}_1})$ with the Legendre transform $\Phi = i(q - \bar{q})$ resulting from (4.3). This choice of Φ does not affect the Legendre-transformed $HCMA$ (4.5), while the transformed differential constraints (4.6), that select particular solutions of $HCMA$ (4.5), become

$$e^{\psi_{\bar{q}}}(\psi_{\bar{q}\bar{q}} + \psi_{q\bar{q}}) = \lambda\psi_{q\bar{z}}, \quad \lambda e^{\psi_q}(\psi_{qq} + \psi_{q\bar{q}}) = \psi_{\bar{q}z}. \tag{4.7}$$

Now, we express $\psi_{q\bar{z}}$ and $\psi_{\bar{q}z}$ from the latter equations and substitute them into $HCMA$ (4.5) with the result

$$\psi_{z\bar{z}} = e^{\psi_q+\psi_{\bar{q}}}(\psi_{qq} + 2\psi_{q\bar{q}} + \psi_{\bar{q}\bar{q}}) \tag{4.8}$$

that can be considered as a linear combination of the three equations (4.5) and (4.7). In the real coordinates x, y in the complex q-plane ($q = x + iy$, $\bar{q} = x - iy$) the equation (4.8) becomes

$$\psi_{z\bar{z}} = e^{\psi_x}\psi_{xx} \tag{4.9}$$

which is the same Boyer-Finley equation (2.3), that we have derived in section 2 by the rotational symmetry reduction, but in the different variables: instead of p we now have $x = (q + \bar{q})/2$. The partner symmetries constraints (4.7) in the real coordinates x, y take the form

$$\psi_{zx} + i\psi_{zy} = 2\lambda \left[e^{(\psi_x - i\psi_y)/2}\right]_x, \quad \psi_{\bar{z}x} - i\psi_{\bar{z}y} = 2\lambda^{-1}\left[e^{(\psi_x + i\psi_y)/2}\right]_x. \tag{4.10}$$

The variable y does not appear explicitly in the Boyer-Finley equation (4.9), being just a parameter, and so it can be regarded as a parameter of a symmetry group of this equation: a change of y will not affect the equation. If ω is the symmetry characteristic of the Boyer-Finley equation

$$\tilde{\psi}_{z\bar{z}} = \exp(\tilde{\psi}_{xx}) \tag{4.11}$$

related to (4.9) by the substitution $\psi = \tilde{\psi}_x$, then the symmetry characteristic of (4.9) is $i\omega_x$, where the constant factor i is introduced for convenience. The Lie equation for the symmetry group with the parameter y and symmetry characteristic $i\omega_x$ reads

$$\psi_y = i\omega_x. \tag{4.12}$$

By eliminating ψ_y in (4.10) with the aid of (4.12) and then integrating the resulting equations with respect to x, we obtain

$$\omega_z = \psi_z - 2\lambda e^{(\psi_x + \omega_x)/2}, \quad \omega_{\bar{z}} = -\psi_{\bar{z}} + 2\lambda^{-1}e^{(\psi_x - \omega_x)/2}. \tag{4.13}$$

These are Bäcklund transformations for the Boyer-Finley equation that we discovered earlier [11]. The differential compatibility condition of the system (4.13) $(\omega_z)_{\bar{z}} = (\omega_{\bar{z}})_z$ reproduces the Boyer-Finley equation (4.9) and the compatibility condition, taken in the form $(\psi_z)_{\bar{z}} = (\psi_{\bar{z}})_z$ yields the determining equation for symmetry characteristics of the Boyer-Finley equation (4.11)

$$\omega_{z\bar{z}} - e^{\psi_x}\omega_{xx} = 0. \tag{4.14}$$

Thus, without any symmetry reduction being done, the Boyer-Finley equation arises as a linear combination of the Legendre-transformed $HCMA$

and differential constraints (4.7) following from the choice of the rotational symmetry for partner symmetries. Furthermore, the differential constraints themselves turn out to be the Bäcklund transformations for the Boyer-Finley equation in a new disguise.

Note what happens if we reverse our procedure. Then starting with the three-dimensional Boyer-Finley equation together with its Bäcklund transformations and considering a symmetry group parameter τ as the fourth coordinate y in the equations, we arrive at the four-dimensional $HCMA$ equation. In this way, partner symmetries provide a lift from three-dimensional noninvariant solutions of the Boyer-Finley equation to four-dimensional noninvariant solutions of $HCMA$ that govern four-dimensional ultra-hyperbolic metrics without Killing vectors.

P. Tod in [15] used invariant solutions to both hyperbolic Boyer-Finley equation and (4.14) for constructing scalar-flat Käler metrics with ultra-hyperbolic signature that admit a symmetry. Earlier C. LeBrun used the elliptic Boyer-Finley equation together with the equation for its symmetries for constructing self-dual metrics with Euclidean signature [16]. Using our Bäcklund transformations, we can obtain new noninvariant solutions of the Boyer-Finley equation, both elliptic and hyperbolic, starting from known symmetries (solutions to (4.14)). This approach was demonstrated in the elliptic case in [11].

5 Lift of noninvariant solutions of the Boyer-Finley equation to $HCMA$

We start with noninvariant solutions to the hyperbolic version of Boyer-Finley equation

$$v_{z\bar{z}} = (e^v)_{xx} \tag{5.1}$$

that we had obtained earlier by the method of group foliation in [8] (noninvariant solutions to the elliptic Boyer-Finley equation were obtained in [7,8]). Those solutions involve a couple of holomorphic and anti-holomorphic functions of one argument $b(z)$ and $\bar{b}(\bar{z})$ that arise as "constants" of integrations. In our construction, the Boyer-Finley equation (4.9) and its solutions depend also on the fourth variable, the parameter y, and hence the integration "constants" in the noninvariant solutions given in [8], b and $\bar{b}$, also should depend

on y:

$$v(x,y,z,\bar{z}) = \ln[x+b(z,y)] + \ln[x+\bar{b}(\bar{z},y)] - 2\ln(z+\bar{z}). \tag{5.2}$$

The Boyer-Finley equations in the forms (4.9) and (5.1) are related to each other by the substitution $v = \psi_x$ and hence solutions of (4.9) are obtained by integrating (5.2) with respect to x with the "constant" of integration $F(z,\bar{z},y)$ that depends on the other three variables:

$$\begin{aligned}\psi \;=\; & [x+b(z,y)]\ln[x+b(z,y)] + [x+\bar{b}(\bar{z},y)]\ln[x+\bar{b}(\bar{z},y)] \\ & - 2x[\ln(z+\bar{z})+1] + F(z,\bar{z},y).\end{aligned} \tag{5.3}$$

The unknown y-dependence in (5.3) is determined by the requirement that ψ should also satisfy the Legendre-transformed $HCMA$ (4.5), since we need solutions of the latter equation.

Thus, we substitute the expression (5.3) for ψ in $HCMA$ (4.5) and, since all the x-dependence is known explicitly, it splits into several equations, corresponding to groups of terms with a different dependence on x. We were able to solve these equations and make a complete analysis of all possible solutions.

List of solutions:

$$\begin{aligned}&\psi = [q+b(z)]\ln[q+b(z)] + [\bar{q}+\bar{b}(\bar{z})]\ln[\bar{q}+\bar{b}(\bar{z})] \\ &-(q+\bar{q})[\ln(z+\bar{z})+1] + \iint \frac{b(z)+\bar{b}(\bar{z})}{(z+\bar{z})^2}\,dz d\bar{z} + r(y),\end{aligned} \tag{5.4}$$

$$\begin{aligned}&\psi = [q+b(z)]\ln[q+b(z)] + [\bar{q}+\bar{b}(\bar{z})]\ln[\bar{q}+\bar{b}(\bar{z})] \\ &-(q+\bar{q})[\ln(z+\bar{z})+1] + \iint \frac{b(z)+\bar{b}(\bar{z})}{(z+\bar{z})^2}\,dz d\bar{z} \\ &+ 2iy\ln\left(\frac{\bar{z}}{z}\right) + r(y),\end{aligned} \tag{5.5}$$

$$\begin{aligned}&\psi = [q+b(z)]\ln[q+b(z)] + [\bar{q}+\bar{b}(\bar{z})]\ln[\bar{q}+\bar{b}(\bar{z})] \\ &-(q+\bar{q})[\ln(z+\bar{z})+1] + \iint \frac{b(z)+\bar{b}(\bar{z})}{(z+\bar{z})^2}\,dz d\bar{z} \\ &+2i\int \ln\left[\frac{\bar{z}+2ik(y)}{z-2ik(y)}\right] dy + r(y).\end{aligned} \tag{5.6}$$

Here $r(y)$ and $k(y)$ are arbitrary smooth real-valued functions of one real variable $y = i(\bar{q}-q)/2$ and $b(z)$ and $\bar{b}(\bar{z})$ are arbitrary holomorphic and

anti-holomorphic functions of one complex variable that arise when the y-dependence of $b(z,y)$ and $\bar{b}(\bar{z},y)$ is completely determined. Solution (5.5) is a particular simple case of the more general solution (5.6) when $k(y)=0$.

Theorem 1 *Up to an arbitrary symmetry transformation of $HCMA$ (4.5), the list of solutions (5.4) - (5.6) is a complete set of solutions of $HCMA$ that can be obtained by lifting noninvariant solutions (5.3) of the Boyer-Finley equation.*

Note that, by construction, we have obtained the solutions of $HCMA$ that satisfy only one additional differential constraint, the Boyer-Finley equation, though in (4.7) we have two constraint equations produced by partner symmetries. If we require that both constraints (4.7) should be satisfied, then we shall obtain a subset of solutions that are invariant with respect to nonlocal symmetries of $HCMA$, though this does not mean invariant solutions in the usual sense [3, 4]. Solutions with such special property are obtained by setting $r(y)$ to be particular linear functions, namely for solution (5.4)

$$r(y)=2(\alpha-\pi)y+r_0 \tag{5.7}$$

and for solutions (5.5) and (5.6)

$$r(y)=2\alpha y+r_0 \tag{5.8}$$

where r_0 is an arbitrary real constant and $\lambda=e^{i\alpha}$ is the constant coefficient in (4.7).

6 Non-invariance of solutions

We have found point symmetries of the Legendre-transformed $HCMA$ (4.5) using computer packages CRACK and LIEPDE by Thomas Wolf [17], being run in the computer algebra system REDUCE 3.8. The symmetry generators are

$$\begin{gathered} X_1=q\partial_q+\bar{q}\partial_{\bar{q}}+(q+\bar{q}+\psi)\partial_\psi, \quad X_2=(q-\bar{q})\partial_\psi \\ X_{a(z)}=a(z)\partial_z-a'(z)q\partial_\psi, \quad X_{c(z)}=c(z)\partial_q, \quad X_{d(z)}=d(z)\partial_\psi \end{gathered} \tag{6.1}$$

together with the complex conjugate generators $\bar{X}_{\bar{a}(\bar{z})}$, $\bar{X}_{\bar{c}(\bar{z})}$, and $\bar{X}_{\bar{d}(\bar{z})}$, where $a(z)$, $c(z)$, and $d(z)$ and their complex conjugates are arbitrary holomorphic and anti-holomorphic functions respectively. Here $\partial_q=\partial/\partial_q$ and so on.

A solution $\psi = f(q, \bar{q}, z, \bar{z})$ is invariant under a one-parameter symmetry Lie group with the generator X if it satisfies the invariance condition

$$X(f - \psi)|_{\psi=f} = 0 \tag{6.2}$$

where, after acting by X on the solution manifold, ψ should be eliminated by using the solution $\psi = f$.

In our problem, a generator of an arbitrary one-dimensional symmetry subgroup, that should be used in the invariance condition (6.2), is a linear combination of the basis generators (6.1)

$$X = C_1 X_1 + C_2 X_2 + X_{a(z)} + \bar{X}_{\bar{a}(\bar{z})} + X_{c(z)} + \bar{X}_{\bar{c}(\bar{z})} + X_{d(z)} + \bar{X}_{\bar{d}(\bar{z})} \tag{6.3}$$

where C_1 and C_2 are arbitrary real constants and constant coefficients of other generators are absorbed in the arbitrary functions.

We apply the invariance condition (6.2) to each of our solutions (5.4)–(5.6) and expect that this condition will either require certain specializations of arbitrary functions $b(z)$, $\bar{b}(\bar{z})$, $r(y)$, and $k(y)$ or give the result that $X = 0$, that is, there is no symmetry of (4.5) with respect to which the solution will be invariant. That would mean that our solutions are generically noninvariant.

With the generator X defined by (6.3), the invariance condition for our first and simplest solution (5.4) has the form (primes denote derivatives)

$$\begin{aligned}
&[C_1 + C_2 - a'(z)]q + [C_1 - C_2 - \bar{a}'(\bar{z})]\bar{q} + d(z) + \bar{d}(\bar{z}) + C_1 \Big\{ b(z) \ln [q + b(z)] \\
&+ \bar{b}(\bar{z}) \ln [\bar{q} + \bar{b}(\bar{z})] - (q + \bar{q})[\ln (z + \bar{z}) - 1] + r(y) + \iint \frac{b(z) + \bar{b}(\bar{z})}{(z + \bar{z})^2}\, dz d\bar{z} \Big\} \\
&\quad = c(z) \ln [q + b(z)] + \bar{c}(\bar{z}) \ln [\bar{q} + \bar{b}(\bar{z})] \qquad (6.4)\\
&\quad - [c(z) + C_1 q][\ln (z + \bar{z}) + (i/2) r'(y)] - [\bar{c}(\bar{z}) + C_1 \bar{q}][\ln (z + \bar{z}) \\
&\quad - (i/2) r'(y)] + a(z) \left\{ b'(z) \Big[\ln \Big(q + b(z) \Big) + 1 \Big] + \int \frac{b(z) + \bar{b}(\bar{z})}{(z + \bar{z})^2} d\bar{z} \right\} \\
&+ \bar{a}(\bar{z}) \left\{ \bar{b}'(\bar{z}) \Big[\ln \Big(\bar{q} + \bar{b}(\bar{z}) \Big) + 1 \Big] + \int \frac{b(z) + \bar{b}(\bar{z})}{(z + \bar{z})^2} dz \right\} - \frac{(q + \bar{q})[a(z) + \bar{a}(\bar{z})]}{z + \bar{z}}
\end{aligned}$$

where $y = i(\bar{q} - q)/2$. Differentiating this equation twice with respect to z and $\bar{z}$ and splitting the resulting equation in q and $\bar{q}$, we arrive at the equation

$$(z + \bar{z})[a'(z) + \bar{a}'(\bar{z})] - 2[a(z) + \bar{a}(\bar{z})] = 0 \tag{6.5}$$

with the simple consequence $a''(z)+\bar{a}''(\bar{z})=0$. After the separation of z and $\bar{z}$, this yields $a''(z)=2i\beta$ and $\bar{a}''(\bar{z})=-2i\beta$, where β is an arbitrary real constant. Obtaining $a(z)$ and $\bar{a}(\bar{z})$ by integration and substituting the result into (6.5), we get

$$a(z)=i\beta z^2+C_3 z+i\gamma, \quad \bar{a}(\bar{z})=-i\beta\bar{z}^2+C_3\bar{z}-i\gamma \tag{6.6}$$

where C_3 and γ are also arbitrary real constants.

Next, we differentiate the invariance condition (6.4) twice, first with respect to q, obtaining

$$\frac{a(z)b'(z)-C_1 b(z)+c(z)}{q+b(z)}-iC_1 r'(y)+(1/4)r''(y)[\bar{c}(\bar{z})-c(z)+C_1(\bar{q}-q)] \\ -\frac{a(z)+\bar{a}(\bar{z})}{z+\bar{z}}+a'(z)-2C_1-C_2=0 \tag{6.7}$$

and then with respect to $\bar{q}$ with the result

$$ir'''(y)[\bar{c}(\bar{z})-c(z)+C_1(\bar{q}-q)]+6C_1 r''(y)=0. \tag{6.8}$$

Differentiating (6.8) with respect to z or $\bar{z}$, we obtain the conditions

$$c'(z)r'''(y)=0, \qquad \bar{c}'(\bar{z})r'''(y)=0 \tag{6.9}$$

that imply the following two cases: $r'''(y)=0$ and $r'''(y)\neq 0$.

Case 1:

$$r'''(y)=0 \quad \Longrightarrow \quad r(y)=\lambda y^2+\mu y+\nu \tag{6.10}$$

with real coefficients. Splitting (6.7) and its complex conjugate in q, $\bar{q}$, we arrive at the relations

$$a(z)b'(z)+c(z)-C_1 b(z)=0, \quad \bar{a}(\bar{z})\bar{b}'(\bar{z})+\bar{c}(\bar{z})-C_1\bar{b}(\bar{z})=0 \tag{6.11}$$

$$(\lambda/2)[\bar{c}(\bar{z})-c(z)]+i\beta(z+\bar{z})-(i\mu+2)C_1-C_2=0 \tag{6.12}$$

where in the last equation we have used the expressions (6.6) for $a(z)$ and $\bar{a}(\bar{z})$, and

$$C_1 r''(y)=0 \qquad \Longleftrightarrow \qquad C_1\lambda=0 \tag{6.13}$$

so that either $C_1=0$ or $\lambda=0$.

Case 1a:

$$\lambda\neq 0, \qquad C_1=0. \tag{6.14}$$

Separating z and $\bar{z}$ in (6.12), we determine $c(z)$, $\bar{c}(\bar{z})$ and C_2 (using that C_2 is real)

$$c(z) = (2i\beta/\lambda)z + c_0, \quad \bar{c}(\bar{z}) = -(2i\beta/\lambda)\bar{z} + \bar{c}_0, \quad C_2 = 0. \tag{6.15}$$

The relations (6.11) become

$$a(z)b'(z) + c(z) = 0, \quad \bar{a}(\bar{z})\bar{b}'(\bar{z}) + \bar{c}(\bar{z}) = 0 \tag{6.16}$$

with $a(z)$, $\bar{a}(\bar{z})$ given by (6.6) and $c(z)$, $\bar{c}(\bar{z})$ by (6.15) respectively, and so they yield the special form of $b(z)$ and $\bar{b}(\bar{z})$. Now, differentiating the invariance condition (6.4) twice with respect to z and $\bar{z}$ and using (6.11), we obtain $\beta = 0, c_0 = 0$ and hence $c(z) = \bar{c}(\bar{z}) = 0$, so that eliminating the trivial case when b and $\bar{b}$ are constants, we conclude that $a(z) = 0$ and $\bar{a}(\bar{z}) = 0$. Then the invariance condition (6.4) reduces to $d(z) + \bar{d}(\bar{z}) = 0$ and thus the symmetry generator (6.3) is zero. Therefore, in the case of quadratic $r(y)$ there is no symmetry with respect to which our solution with the nonconstant b and $\bar{b}$ could be invariant.

Case 1b:

$$\lambda = 0, \qquad C_1 \neq 0 \quad \Longrightarrow \quad r(y) = \mu y + \nu. \tag{6.17}$$

The relation (6.12) in this case is split in z and $\bar{z}$ to give

$$\beta = 0, \qquad \mu = 0 \quad \text{or} \quad C_1 = 0, \qquad C_2 = -2C_1 \tag{6.18}$$

because C_1 and C_2 are real. If $C_1 = 0$, we are back to Case 1a, so $\mu = 0$ and from (6.17) $r(y) = \nu$ should be constant for an invariant solution. Invariance condition (6.4) with the use of the relations (6.11), after splitting in $\bar{q}$, yields $C_1 = 0$, so we are again back to the Case 1a.

Case 2:

$$r'''(y) \neq 0 \quad \Longrightarrow \quad c'(z) = 0, \quad \bar{c}'(\bar{z}) = 0 \tag{6.19}$$

where we have used (6.9), so $c(z) = c$ and $\bar{c}(\bar{z}) = \bar{c}$ are now constants.

Case 2a: $C_1 \neq 0$.

Then (6.8) is easily integrated to yield

$$r(y) = \frac{r_0}{8C_1^2[2C_1 y + i(\bar{c} - c)]} + r_1 y + r_2 \tag{6.20}$$

where r_0, r_1 and r_2 are constants of integrations. Substituting the expression (6.20) into invariance condition (6.4) and splitting the resulting equation in q and $\bar{q}$, we obtain

$$r_0 = 0, \quad \beta = 0, \quad C_1 = 0, \quad C_2 = 0 \tag{6.21}$$

that contradicts the assumption of the Case 2a.

Case 2b: $C_1 = 0$.

Then (6.8) yields $\bar{c} = c$ and splitting (6.7) in q and the complex conjugate to (6.7) in $\bar{q}$, we obtain

$$a(z)b'(z) = -c, \quad \bar{a}(\bar{z})\bar{b}'(\bar{z}) = -c, \quad \beta = 0, \quad C_2 = 0 \tag{6.22}$$

so that a and $\bar{a}$ are linear functions

$$a(z) = C_3 z + i\gamma, \quad \bar{a}(\bar{z}) = C_3 \bar{z} - i\gamma. \tag{6.23}$$

The invariance condition (6.4) simplifies to

$$\begin{aligned} d(z) + \bar{d}(\bar{z}) &= -2c[\ln(z+\bar{z}) + 1] \\ &+ a(z)\int \frac{b(z) + \bar{b}(\bar{z})}{(z+\bar{z})^2}\, d\bar{z} + \bar{a}(\bar{z})\int \frac{b(z) + \bar{b}(\bar{z})}{(z+\bar{z})^2} dz. \end{aligned} \tag{6.24}$$

The term with the logarithm cannot be compensated by the integral terms. Indeed, the only possibility for the integrals to produce $\ln(z+\bar{z})$ is when $b = kz$, $\bar{b} = k\bar{z}$ with the real constant k while $a, \bar{a}$ are constant (at $C_3 = 0$), but then the integral terms cancel each other. Therefore, the coefficient of the logarithm should vanish, so $c = 0$ and the relations (6.22) yield $a = \bar{a} = 0$ for nonconstant $b(z)$ and $\bar{b}(\bar{z})$. It follows then from (6.24) that $d(z) + \bar{d}(\bar{z}) = 0$ and the symmetry generator X in (6.3) vanishes. Thus, apart from the case of constant b and $\bar{b}$, there is no symmetry under which our solution (5.4) would be invariant.

For our more complicated solutions (5.5) and (5.6)), it is obvious that invariance conditions would be even more difficult to satisfy and hence we can summarize our results as follows.

Theorem 2 *If the functions $b(z)$, $\bar{b}(\bar{z})$ are not constants, the formulas (5.4) - (5.6) yield noninvariant solutions of HCMA (4.5).*

Note that for non-invariance of solution (5.4) the condition of the theorem 2 is necessary and sufficient.

As a consequence, the ultra-hyperbolic metrics governed by the potentials ψ in (5.4) - (5.6), constructed in the next sections, have no symmetries (Killing vectors).

7 Ultra-hyperbolic hyper-Kähler metrics, Newman-Penrose co-frame and Legendre transformation

Four-dimensional hyper-Kähler metrics

$$ds^2 = u_{1\bar{1}}dz^1 d\bar{z}^1 + u_{1\bar{2}}dz^1 d\bar{z}^2 + u_{2\bar{1}}dz^2 d\bar{z}^1 + u_{2\bar{2}}dz^2 d\bar{z}^2 \tag{7.1}$$

satisfy Einstein field equations with either Euclidean or ultra-hyperbolic signature, if the Kähler potential u satisfies elliptic (1.1) or hyperbolic (1.2) complex Monge-Ampère equation respectively [1]. Such metrics are Ricci-flat and have (anti-)self-dual curvature. Here we restrict ourselves to the $HCMA$ equation (1.2) and hence the metric (7.1) has ultra-hyperbolic signature. This becomes obvious if we use the tetrad of Newman-Penrose moving co-frame $\{l, \bar{l}, m, \bar{m}\}$ [18, 19] corresponding to the metric (7.1)

$$l = \frac{1}{\sqrt{u_{1\bar{1}}}}(u_{\bar{1}1}dz^1 + u_{\bar{1}2}dz^2), \qquad m = \frac{1}{\sqrt{u_{1\bar{1}}}}\,dz^2 \tag{7.2}$$

where $\bar{l}$ and $\bar{m}$ are complex conjugates to l and m. Indeed, if we express $u_{2\bar{2}}$ from the equation (1.2) and substitute this in the metric (7.1), then the metric can be written in the form

$$ds^2 = l \otimes \bar{l} - m \otimes \bar{m} \tag{7.3}$$

so that the signature of the metric is ultra-hyperbolic $(+ + - -)$. The Newman-Penrose co-frame provides most convenient way of calculating Riemann curvature two-forms.

Because of the discrete symmetry $1 \leftrightarrow 2, \bar{1} \leftrightarrow \bar{2}$ of $HCMA$ and the metric (7.1), another possible co-frame tetrad is obtained from (7.2) by this discrete transformation

$$l' = \frac{1}{\sqrt{u_{2\bar{2}}}}(u_{\bar{2}1}dz^1 + u_{\bar{2}2}dz^2), \qquad m' = \frac{1}{\sqrt{u_{2\bar{2}}}}\,dz^1 \tag{7.4}$$

and it also satisfies the relation (7.3)

$$ds^2 = l' \otimes \bar{l}' - m' \otimes \bar{m}'. \tag{7.5}$$

Since we have exact solutions of $HCMA$ that was subjected to a combination of a point and Legendre transformation, in order to use these solutions,

we have to perform the same transformations upon the metric (7.1) and the Newman-Penrose co-frame (7.2).

The point transformation $z_1 = e^{\zeta_1}$, $z_2 = e^{\zeta_2}$ leaves the metric form-invariant

$$ds^2 = u_{\zeta_1\bar\zeta_1} d\zeta^1 d\bar\zeta^1 + u_{\zeta_1\bar 2} d\zeta^1 d\bar z^2 + u_{2\bar\zeta_1} dz^2 d\bar\zeta^1 + u_{2\bar 2} dz^2 d\bar z^2. \tag{7.6}$$

The tetrad 1-forms become

$$l = \frac{u_{\zeta_1\bar\zeta_1} d\zeta^1 + u_{2\bar\zeta_1} dz^2}{\sqrt{u_{\zeta_1\bar\zeta_1}}}, \qquad m = \frac{e^{(\zeta_1+\bar\zeta_1)/2} dz^2}{\sqrt{u_{\zeta_1\bar\zeta_1}}} \tag{7.7}$$

together with their complex conjugates $\bar l, \bar m$, where we have skipped the exponential factors $e^{(\zeta_1-\bar\zeta_1)/2}$ and $e^{(\bar\zeta_1-\zeta_1)/2}$ in l and $\bar l$ since they cancel each other in the formula (7.3) for the metric.

Next we perform the Legendre transformation (4.3) of the metric and moving co-frame. The metric becomes

$$\begin{aligned} ds^2 &= \frac{-1}{\Delta_-}\Big\{\psi_{qq}(\psi_{\bar q\bar q} dq + \psi_{\bar q z} dz)^2 + \psi_{\bar q\bar q}(\psi_{q\bar q} d\bar q + \psi_{q\bar z} d\bar z)^2 \\ &+ \Delta_+(\psi_{q\bar q} dq d\bar q + \psi_{q\bar z} dq d\bar z + \psi_{\bar q z} d\bar q dz + \psi_{z\bar z} dz d\bar z) \\ &+ 2\psi_{q\bar q}(\psi_{q\bar z}\psi_{\bar q z} - \psi_{q\bar q}\psi_{z\bar z}) dz d\bar z\Big\} \end{aligned} \tag{7.8}$$

where $\Delta_- = \psi_{qq}\psi_{\bar q\bar q} - \psi_{q\bar q}^2$, $\Delta_+ = \psi_{qq}\psi_{\bar q\bar q} + \psi_{q\bar q}^2$ and $\psi(q, \bar q, z, \bar z)$ has to satisfy (4.5), the Legendre transform of $HCMA$. The Legendre transform of the moving co-frame is

$$\begin{aligned} l &= \frac{\psi_{q\bar q}(\psi_{qq} dq + \psi_{q\bar q} d\bar q + \psi_{q\bar z} d\bar z) + \psi_{qq}\psi_{\bar q z} dz}{\sqrt{-\psi_{q\bar q}\Delta_-}} \\ m &= e^{(\psi_q+\psi_{\bar q})/2}\sqrt{\frac{-\Delta_-}{\psi_{q\bar q}}}\, dz \end{aligned} \tag{7.9}$$

together with their complex conjugates. It is easy to check that these ds^2, l, and m together with $\bar l$ and $\bar m$ still satisfy the relation (7.3). The Legendre transform of the co-frame (7.4) is

$$\begin{aligned} l' &= \Big\{(\psi_{q\bar q}\psi_{\bar q\bar z} - \psi_{\bar q\bar q}\psi_{q\bar z})(\psi_{qq} dq + \psi_{q\bar q} d\bar q + \psi_{q\bar z} d\bar z) \\ &\quad + [\psi_{\bar q z}(\psi_{qq}\psi_{\bar q\bar z} - \psi_{q\bar q}\psi_{q\bar z}) - \psi_{z\bar z}\Delta_-] dz\Big\} \times \\ &\{\Delta_-[\psi_{qz}(\psi_{q\bar z}\psi_{\bar q\bar q} - \psi_{q\bar q}\psi_{\bar q\bar z}) + \psi_{\bar q z}(\psi_{qq}\psi_{\bar q\bar z} - \psi_{q\bar q}\psi_{q\bar z}) - \psi_{z\bar z}\Delta_-]\}^{-1/2} \\ m' &= e^{\psi_q}\sqrt{\Delta_-}\,(\psi_{qq} dq + \psi_{q\bar q} d\bar q + \psi_{qz} dz + \psi_{q\bar z} d\bar z) \times \\ &\{\psi_{qz}(\psi_{q\bar z}\psi_{\bar q\bar q} - \psi_{q\bar q}\psi_{\bar q\bar z}) + \psi_{\bar q z}(\psi_{qq}\psi_{\bar q\bar z} - \psi_{q\bar q}\psi_{q\bar z}) - \psi_{z\bar z}\Delta_-\}^{-1/2} \end{aligned} \tag{7.10}$$

and their complex conjugates. These l', m', $\bar{l}'$, $\bar{m}'$, and ds^2 satisfy the relation (7.5).

8 New ultra-hyperbolic metrics and moving co-frames

To obtain new ultra-hyperbolic Ricci-flat metrics without Killing vectors together with moving co-frames, we use for ψ in the formulas (7.8) and (7.9) our noninvariant solutions of $HCMA$ (4.5) from the list (5.4) - (5.6).

For the first solution (5.4) from this list, the metric takes the form

$$ds^2 = \frac{-4}{(z+\bar{z})^2[(q+\bar{q}+b+\bar{b})r''(y)+4]}\Big\{(\sqrt{A}dq-\sqrt{D}dz)^2 \tag{8.1}$$
$$+(\sqrt{\bar{A}}d\bar{q}-\sqrt{\bar{D}}d\bar{z})^2+B\left[dqd\bar{q}\pm\sqrt{D/A}(dqd\bar{z}+d\bar{q}dz)\right]+Edzd\bar{z}\Big\}$$

where the plus or minus sign corresponds to $r''(y)>0$ or $r''(y)<0$ respectively and the metric coefficients are defined by the formulas

$$D=\frac{1}{4}\left(\bar{q}+\bar{b}\right)\left[(q+b)\,r''(y)+4\right],\quad A=\frac{1}{16}\,(z+\bar{z})^2(r''(y))^2D$$

together with their complex conjugates and

$$B=-\frac{1}{32}\,(z+\bar{z})^2\,r''(y)\left[(q+b)\left(\bar{q}+\bar{b}\right)(r''(y))^2\right.$$
$$\left.+2\left(q+\bar{q}+b+\bar{b}\right)r''(y)+8\right]$$

$$E=\frac{1}{4}\left[(q+b)^2+(\bar{q}+\bar{b})^2\right]r''(y)+q+\bar{q}+b+\bar{b}.$$

From now on, $b=b(z)$ and $\bar{b}=\bar{b}(\bar{z})$ are arbitrary holomorphic and anti-holomorphic functions of one complex argument and $r(y)$ is an arbitrary real-valued function of one real variable $y=i(\bar{q}-q)/2$.

The calculation of the affine connection one-forms and the curvature two-forms is greatly facilitated by the use of the Newman-Penrose moving co-frame [18, 19]. For generic solutions we shall use the first, simpler co-frame $l,\bar{l},m,\bar{m}$ defined by (7.9).

The co-frame forms for the first solution are

$$l = \frac{(z+\bar{z})(q+b)(r''(y))^2(d\bar{q}-dq)+4(q+b)r''(y)(d\bar{z}-dz)-(z+\bar{z})r''(y)dq-dz}{4(z+\bar{z})\{r''(y)[(q+\bar{q}+b+\bar{b})r''(y)+4]\}^{1/2}}$$

$$m = \left\{\frac{(q+\bar{q}+b+\bar{b})r''(y)+4}{r''(y)}\right\}^{1/2} \frac{dz}{z+\bar{z}} \tag{8.2}$$

and the complex conjugates $\bar{l}, \bar{m}$.

For the second solution (5.5), the metric becomes

$$\begin{aligned} ds^2 = {} & 4\frac{(\sqrt{A}dq-\sqrt{D}dz)^2+(\sqrt{\bar{A}}d\bar{q}-\sqrt{\bar{D}}d\bar{z})^2}{z^2\bar{z}^2(z+\bar{z})^2[4-(q+\bar{q}+b+\bar{b})r''(y)]} \\ & + 4B\frac{[z(z+\bar{z})r''(y)dq+4\bar{z}dz][\bar{z}(z+\bar{z})r''(y)d\bar{q}+4zd\bar{z}]}{z^3\bar{z}^3(z+\bar{z})^4(r''(y))^2[4-(q+\bar{q}+b+\bar{b})r''(y)]} \\ & + \frac{[4-(q+\bar{q}+b+\bar{b})r''(y)]}{(z+\bar{z})^2r''(y)}\, dzd\bar{z} \end{aligned} \tag{8.3}$$

where the metric coefficients are defined by the formulas

$$D = \frac{1}{4}\,\bar{z}^4(\bar{q}+\bar{b})[(q+b)r''(y)-4], \quad A = \left(\frac{z}{4\bar{z}}\right)^2 (z+\bar{z})^2(r''(y))^2 D$$

together with their complex conjugates and

$$B = -\frac{z^2\bar{z}^2}{32}\,(z+\bar{z})^2r''(y)[(q+b)(\bar{q}+\bar{b})(r''(y))^2-2(q+\bar{q}+b+\bar{b})r''(y)+8].$$

The co-frame tetrad becomes

$$\begin{aligned} l = {} & [4z\bar{z}(z+\bar{z})]^{-1}\left\{\frac{\bar{q}+\bar{b}}{(q+b)r''(y)[(q+\bar{q}+b+\bar{b})r''(y)-4]}\right\}^{1/2} \times \\ & \Big\{z\bar{z}(z+\bar{z})r''(y)[(q+b)r''(y)(dq-d\bar{q})-4dq] \\ & -4(q+b)r''(y)(z^2d\bar{z}-\bar{z}^2dz)-16\bar{z}^2dz\Big\} \\ m = {} & \left\{\frac{(q+\bar{q}+b+\bar{b})r''(y)-4}{r''(y)}\right\}^{1/2} \frac{dz}{z+\bar{z}} \end{aligned} \tag{8.4}$$

together with $\bar{l}$, $\bar{m}$.

For the third solution (5.6), generalizing (5.5), we use a shorthand notation

$$
\begin{aligned}
V &= (z+\bar{z})k'(y) - \frac{1}{4}\,(z-2ik(y))(\bar{z}+2ik(y))r''(y) \\
W &= (q+b)V + (z-2ik(y))(\bar{z}+2ik(y))
\end{aligned}
\tag{8.5}
$$

and $\bar{W}$ is the complex conjugate to W. Here $k = k(y)$ and $r(y)$ are arbitrary smooth real-valued functions of a real variable $y = i(\bar{q}-q)/2$ that appear in the third solution (5.6). The metric has the form

$$
\begin{aligned}
&ds^2 = \\
&-\Big\{(z+\bar{z})^2(z-2ik)^2(\bar{z}+2ik)^2[(q+\bar{q}+b+\bar{b})V+(z-2ik)(\bar{z}+2ik)]\Big\}^{-1} \\
&\quad\times\Big\{(\bar{q}+\bar{b})W\Big[(\bar{z}+2ik)^2dz-(z+\bar{z})V\,dq\Big]^2 \\
&\quad + (q+b)\bar{W}\Big[(z-2ik)^2d\bar{z}-(z+\bar{z})V\,d\bar{q}\Big]^2 + \Big[W\bar{W}+(q+b)(\bar{q}+\bar{b})V^2\Big] \\
&\quad\times\Big[-(z+\bar{z})^2V\,dq d\bar{q} + (z+\bar{z})\Big((z-2ik)^2dq d\bar{z}+(\bar{z}+2ik)^2 d\bar{q} dz\Big) \\
&\quad+(q+\bar{q}+b+\bar{b})(z-2ik)(\bar{z}+2ik)dz d\bar{z}\Big]\Big\} \\
&\quad+\frac{2(q+b)(\bar{q}+\bar{b})V}{(z+\bar{z})^2(z-2ik)(\bar{z}+2ik)}\,dz d\bar{z}
\end{aligned}
\tag{8.6}
$$

and the co-frame 1-forms are

$$
\begin{aligned}
l &= \frac{(z+\bar{z})V\Big[\bar{W}d\bar{q}-(\bar{q}+\bar{b})V\,dq\Big]+(\bar{q}+\bar{b})(\bar{z}+2ik)^2V\,dz-(z-2ik)^2\bar{W}d\bar{z}}{(z+\bar{z})(\bar{z}+2ik)^2\{V[(q+\bar{q}+b+\bar{b})V+(z-2ik)(\bar{z}+2ik)]\}^{1/2}} \\
m &= \left\{\frac{(q+\bar{q}+b+\bar{b})V+(z-2ik)(\bar{z}+2ik)}{V}\right\}^{1/2}\,\frac{dz}{z+\bar{z}}
\end{aligned}
\tag{8.7}
$$

and the complex conjugates $\bar{l}$, $\bar{m}$.

By utilizing the moving co-frames, we were able to compute Riemann curvature two-forms for our solutions using the package EXCALC (Exterior Calculus of Modern Differential Geometry) [12] in the computer algebra system REDUCE 3.8 [13].

The special solutions (5.4) and (5.5) meeting the restrictions (5.7) and (5.8) respectively, that satisfy both constraints (4.7), are simple enough to enable us to present explicitly metrics, moving co-frames, and Riemann curvature tensors. For the first special solution (5.4) with the restriction (5.7),

the metric reads

$$ds^2 = (z+\bar{z})^{-2}\{(z+\bar{z})(dqd\bar{z}+d\bar{q}dz) - \Big[\Big(q+b(z)\Big)d\bar{z} + \Big(\bar{q}+\bar{b}(\bar{z})\Big)dz\Big](dz+d\bar{z})\}. \tag{8.8}$$

For the second special solution (5.5) with the restriction (5.8), the metric is

$$\begin{aligned} ds^2 = [z\bar{z}(z+\bar{z})]^{-2}\Big\{\Big[\Big(\bar{q}+\bar{b}(\bar{z})\Big)\bar{z}^2dz + \Big(q+b(z)\Big)z^2d\bar{z}\Big]\Big(\bar{z}^2dz + z^2d\bar{z}\Big) \\ + z\bar{z}(z+\bar{z})\Big(z^2dqd\bar{z} + \bar{z}^2d\bar{q}dz\Big)\Big\}. \end{aligned} \tag{8.9}$$

Both of these metrics are Ricci-flat and have only one non-vanishing component of the Riemann curvature tensor

$$R_{3434} = 2^{-1}(z+\bar{z})^{-3}\Big\{2\Big[b'(z)+\bar{b}'(\bar{z})\Big] - (z+\bar{z})\Big[b''(z)+\bar{b}''(\bar{z})\Big]\Big\} \tag{8.10}$$

for the first special solution and

$$\begin{aligned} R_{3434} = \Big(2z^2\bar{z}^2(z+\bar{z})^3\Big)^{-1}\Big\{(z+\bar{z})\Big[z^4b''(z)+\bar{z}^4\bar{b}''(\bar{z})\Big] \\ + 2z^3(z+2\bar{z})b'(z) + 2\bar{z}^3(\bar{z}+2z)\bar{b}'(\bar{z})\Big\} \end{aligned} \tag{8.11}$$

for the second special solution. For the Riemann tensor ${R^a}_{bcd}$ there are two non-vanishing components

$${R^2}_{434} = -{R^1}_{334} = 2(z+\bar{z})R_{3434}$$

for the first special solution and

$${R^2}_{434} = \frac{2z}{\bar{z}}\,R_{3434}, \quad {R^1}_{334} = \frac{2\bar{z}}{z}\,R_{3434}$$

for the second special solution.

For the two special solutions the first moving co-frame (7.9) becomes singular because of the vanishing $r''(y)$ (and hence $\psi_{q\bar{q}}$) in the denominators. There is no such difficulty for the more general solution (5.6) with the restriction (5.8). Therefore, for the two special solutions we have to use the second co-frame $l', \bar{l}', m', \bar{m}'$ defined by (7.10). For the first special solution it becomes

$$\begin{aligned} l' &= (z+\bar{z})^{-3/2}[-(b'+\bar{b}')]^{-1/2}\{(z+\bar{z})(dq-\bar{b}'dz) - (q+b)(dz+d\bar{z})\} \\ m' &= (z+\bar{z})^{-3/2}[-(b'+\bar{b}')]^{-1/2}[(q+b)(dz+d\bar{z}) - (z+\bar{z})(dq+b'dz)] \end{aligned} \tag{8.12}$$

together with complex conjugates. For the second special solution the second moving co-frame reads

$$\begin{aligned}
l' &= \Big\{(q+b)z(\bar{z}^2dz + z^2d\bar{z}) + \bar{z}(z+\bar{z})\{[2\bar{z}(\bar{q}+\bar{b}) - \bar{z}^2\bar{b}']dz + z^2dq\}\Big\} \times \\
&\quad z^{-1/2}[\bar{z}(z+\bar{z})]^{-3/2}\{2[z(q+b)+\bar{z}(\bar{q}+\bar{b})] - (z^2b' + \bar{z}^2\bar{b}')\} \\
m' &= \sqrt{\bar{z}}[z(z+\bar{z})]^{-3/2}\{(q+b)[(2z+\bar{z})\bar{z}dz - z^2d\bar{z}] - z\bar{z}(z+\bar{z})(dq + b'dz)\} \\
&\quad \times\{2[z(q+b)+\bar{z}(\bar{q}+\bar{b})] - (z^2b' + \bar{z}^2\bar{b}')\} \qquad (8.13)
\end{aligned}$$

and their complex conjugates.

Using these co-frames with the package EXCALC we were able to compute Riemann curvature two-forms for both solutions. For the first special solution they read

$$\begin{aligned}
&R^2{}_2 = \frac{2}{b'+\bar{b}'}\left(\frac{b''+\bar{b}''}{b'+\bar{b}'} - \frac{2}{z+\bar{z}}\right) \times \\
&[\mathrm{o}(2)\wedge\mathrm{o}(3) - \mathrm{o}(1)\wedge\mathrm{o}(4) - \mathrm{o}(3)\wedge\mathrm{o}(4) - \mathrm{o}(1)\wedge\mathrm{o}(2)] \\
&R^3{}_1 = R^3{}_3 = R^2{}_4 = -R^1{}_1 = -R^4{}_2 = -R^1{}_3 = -R^4{}_4 = R^2{}_2 \\
&R^1{}_2 = R^1{}_4 = R^2{}_1 = R^2{}_3 = R^3{}_2 = R^3{}_4 = R^4{}_1 = R^4{}_3 = 0. \qquad (8.14)
\end{aligned}$$

From now on we use the notation $\mathrm{o}(1) = l'$, $\mathrm{o}(2) = \bar{l}'$, $\mathrm{o}(3) = m'$, and $\mathrm{o}(4) = \bar{m}'$ for the co-frame tetrads.

For the second special solution Riemann curvature two-forms are

$$\begin{aligned}
&R^1{}_1 = (z\bar{z})^{-2}(z+\bar{z})^{-1}\Big\{z^2b' + \bar{z}^2\bar{b}' - 2[z(q+b)+\bar{z}(\bar{q}+\bar{b})]\Big\}^{-2} \times \\
&\Big[(z+\bar{z})(z^4b'' + \bar{z}^4\bar{b}'') + 2z^3(z+2\bar{z})b' + 2\bar{z}^3(2z+\bar{z})\bar{b}'\Big] \times \qquad (8.15) \\
&\Big\{z^4\,\mathrm{o}(2)\wedge\mathrm{o}(3) - \bar{z}^4\,\mathrm{o}(1)\wedge\mathrm{o}(4) - (z\bar{z})^2[\mathrm{o}(1)\wedge\mathrm{o}(2) + \mathrm{o}(3)\wedge\mathrm{o}(4)]\Big\} \\
&R^2{}_2 = R^3{}_3 = R^4{}_4 = -R^1{}_1,\ R^1{}_3 = R^4{}_2 = \frac{z^2}{\bar{z}^2}R^1{}_1,\ R^2{}_4 = -R^3{}_1 = \frac{\bar{z}^2}{z^2}R^1{}_1 \\
&R^1{}_2 = R^1{}_4 = R^2{}_1 = R^2{}_3 = R^3{}_2 = R^3{}_4 = R^4{}_1 = R^4{}_3 = 0.
\end{aligned}$$

9 Conclusions

Our goal is to obtain noninvariant solutions of four-dimensional heavenly equations because they will yield new gravitational metrics with no Killing vectors. In particular, this property characterizes the famous gravitational

instanton $K3$ where the metric potential should be a noninvariant solution of the elliptic complex Monge-Ampère equation. In this paper we have developed a suitable approach for solving similar problem for an easier case of the hyperbolic complex Monge-Ampère equation. This approach is based on the use of partner symmetries for lifting noninvariant solutions of three-dimensional equations, that can be obtained from $HCMA$ by the symmetry reduction, to non-invariant solutions of the original four-dimensional equation.

A symmetry reduction of a partial differential equation reduces by one the number of independent variables in the original equation, so that the reduced equation is easier to solve. Its solutions are solutions of the original PDE that are invariant under the symmetry that was used in the reduction. Even if we found noninvariant solutions of the reduced equation, it would only mean that no further symmetry reduction was being made and they would still be invariant solutions of the original equation. On an example of the hyperbolic complex Monge-Ampère equation, we have shown that partner symmetries, when they exist, provide a possibility for a procedure reverse to the symmetry reduction: a lift of noninvariant solutions of the reduced equation to noninvariant solutions of the original equation of higher dimensions. We have developed such a procedure for $HCMA$ and obtained new noninvariant solutions of this equation. Using these solutions as metric potentials, we have obtained new gravitational metrics with the ultra-hyperbolic signature that have no Killing vectors. The calculation of the affine connection one-forms and the curvature two-forms is greatly facilitated by the use of the Newman-Penrose moving co-frame which we have calculated for all our solutions.

We are now in the process of developing a modified lifting procedure to apply it to the elliptic complex Monge-Ampère equation. Using new noninvariant solutions of this equation as metric potentials, we shall obtain new gravitational metrics with the Euclidean signature and with no Killing vectors. We hope to obtain in such a way at least some pieces of the Kummer surface $K3$.

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